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Three-point functions in $\mathcal{N} = 4$ Yang-Mills theory and pp-waves

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Abstract: Recently it has been proposed that the coefficient of the three-point function of the BMN operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is related to the three-string interactions in the pp-wave background. We calculate three-point functions of these operators to the first order in the effective Yang-Mills coupling $\lambda' = g^2_{\mathrm{YM}} N/J^2$ in planar perturbation theory. On the string theory side, we derive the explicit expressions of the Neumann matrices to all orders in $1/(\mu p^+ \alpha')^2$. This allows us to compute the corresponding three-string scattering amplitudes. This provides an all orders prediction for the field theory three-point functions. We compare our field theory results with the string theory results to the subleading order in $1/(\mu p^+ \alpha')^2$ and find perfect agreement.

Keywords: String Field Theory, Penrose limit and pp-wave background.

Contents

1. Introduction

Berenstein, Maldacena and Nastase (BMN) put forward an insightful proposal towards understanding of the massive string modes [1]. The field theory side of this correspondence is the conformal $\mathcal{N} = 4$ SYM in a new double scaling limit. The nature of this double scaling limit was further elucidated in [2, 3, 4].

Crucial to the BMN line of reasoning is the emergence of the pp-wave background, which arises as a Penrose limit [6] of $AdS_5 \times S^5$ [7, 8, 9]. This background is maximally supersymmetric and remarkably string theory in this background is exactly solvable, see [10, 11] for closed strings, and [12, 13, 14] for the open string case. First attempts towards understanding of the holographic relation in the pp-wave context were made in [15, 16, 17, 18]. The pp-wave/SYM correspondence of BMN goes significantly beyond the original AdS/CFT duality [19]–[23] since it provides a map between field theory operators and generic string states in the lightcone gauge, and not just the supergravity multiplet ($n = 0$). For example,

$$\frac{1}{\sqrt{J} N^{J/2+1}} \operatorname{Tr} Z^J \longleftrightarrow |0, p^+\rangle, \tag{1.1}$$

$$\frac{1}{\sqrt{J} N^{J/2+1}} \sum_{l=0}^{J} \operatorname{Tr}[\phi^3 Z^l \phi^4 Z^{J-l}] e^{\frac{2\pi i n l}{J}} \longleftrightarrow a_n^{7\dagger} a_{-n}^{8\dagger} |0, p^+\rangle \,. \tag{1.2}$$

These relations give the string vacuum and the first excited string states in terms of SYM operators. Here N is the number of colours and $Z = \phi_5 + i\phi_6$ is a complex scalar field with unit R charge. The scaling dimensions Δ of the Yang-Mills operators in (1.1), (1.2) are related to the masses of the corresponding string states via

$$\Delta - J = \frac{H_{\rm lc}}{\mu} \,, \tag{1.3}$$

where $H_{\rm lc}$ is the lightcone string hamiltonian and μ is the scale of the pp-wave metric. Written in terms of gauge theory parameters, this gives a prediction for the conformal dimension of the BMN operators

$$\Delta - J = \sqrt{1 + \frac{g_{\rm YM}^2 N n^2}{J}}\,. \tag{1.4}$$

The BMN correspondence is understood to hold in the double scaling limit:

$$N \to \infty\,, \qquad J \sim \sqrt{N} \qquad \text{with } g_{\rm YM} \text{ fixed.} \tag{1.5}$$

In this limit the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ is infinite and perturbative calculations in gauge theory are hopeless in general. A well known exception to this rule concerns BPS operators which receive no perturbative corrections at all, and their scaling dimensions Δ coincide with their free theory engineering dimensions Δ^0. BMN instead considered a class of 'near-BPS' operators, as in (1.2) with large R-charge J, which do receive quantum corrections, but these operators are constructed in such a way that in perturbative evaluation of their scaling dimensions, λ is accompanied by a suppression factor $1/J^2$. For these operators the coupling is effectively

$$\lambda' = \frac{g_{\rm YM}^2 N}{J^2} = \frac{1}{(\mu p^+ \alpha')^2}\,, \tag{1.6}$$

which is finite in the large N limit (1.5) and can be taken small [1], see also [2, 3, 4].

It was further assumed in [1] that the gauge theory remains planar in the limit (1.5) for the class of BMN operators even though the original 't Hooft coupling λ is infinite. Non-planar diagrams in the BMN limit (1.5) were first studied in [2] and in [3] and were found to be important and governed by J^4/N^2. It follows from the double scaling limit (1.5) that in addition to λ' defined in (1.6), there is a second dimensionless constant

$$g_2 := \frac{J^2}{N} = 4\pi g_s (\mu p^+ \alpha')^2\,, \tag{1.7}$$

which plays the rôle of the genus counting parameter as explained in [2, 3].

In this paper, we will be mostly concerned with the following BMN operators:

$$\mathcal{O}^J_{\rm vac} := \frac{1}{\sqrt{JN^J}}\,{\rm tr}(Z^J), \tag{1.8}$$

$$\mathcal{O}^J_0 := \frac{1}{\sqrt{N^{J+1}}}\,{\rm tr}(\Phi Z^J), \tag{1.9}$$

$$\mathcal{O}^J_{n,-n} := \frac{1}{\sqrt{JN^{J+2}}} \sum_{l=0}^{J} {\rm tr}(\Phi Z^l \Psi Z^{J-l}) e^{\frac{2\pi i n l}{J}} \tag{1.10}$$

and their two and three-point correlation functions. Here

$$\Phi = \phi_1 + i\phi_2\,, \qquad \Psi = \phi_3 + i\phi_4\,, \qquad Z = \phi_5 + i\phi_6 \tag{1.11}$$

are the three complex scalar fields of the $\mathcal{N} = 4$ theory. In perturbation theory, the flavour of Φ, Ψ and Z is conserved. The operators $\mathcal{O}^J_{\rm vac}, \mathcal{O}^J_0, \mathcal{O}^J_{0,0}$ are half BPS and correspond to the string vacuum and supergravity states. The operator $\mathcal{O}^J_{n,-n}$ for $n \neq 0$ is non-BPS and receives quantum corrections to its scaling dimension. At the level of planar diagrams, BMN operators do not mix [2, 3, 4] and have well-defined conformal dimensions. Two and three-point functions of chiral operators with arbitrary R-charges had been calculated in [5] and [2]. It follows from the conformal invariance of the theory that the two-point function can be written in the canonical form[1]

$$\langle \mathcal{O}_i(0)\mathcal{O}_j(x)\rangle = \frac{\delta_{ij}}{(4\pi^2 x^2)^{\Delta_i}}\,. \tag{1.12}$$

Furthermore conformal invariance implies that the three-point function takes the form

$$\langle \mathcal{O}_{i_1}(x_1)\mathcal{O}_{i_2}(x_2)\mathcal{O}_{i_3}(x_3)\rangle = \frac{C_{i_1 i_2 i_3}}{(4\pi^2 x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(4\pi^2 x_{13}^2)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}(4\pi^2 x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}} \tag{1.13}$$

where $x_{ij}^2 := (x_i - x_j)^2$. When nonplanar diagrams are taken into account, BMN operators $\mathcal{O}^J_{n,-n}$ with different nonvanishing values of n mix with each other already in free field theory [2, 3]. Hence the original BMN operators do not have well defined conformal dimensions and one has to define a new basis of such operators which does not mix [2]. This redefinition has to be implemented order by order in g_2 and λ'. Equations (1.12) and (1.13) represent the correlation functions of these redefined operators with well-defined conformal dimensions Δ_i. The authors of [4] calculated anomalous dimensions to the order λ'^2 at the planar level (leading order in g_2). Alternatively, one may also go beyond the original BMN perturbative computation of anomalous dimensions by including higher genus diagrams [2, 3]. Planar three-point functions involving nonchiral operators in free field theory were calculated in [3]. In this paper, we will consider the planar limit in order to work with the original BMN basis of operators, thus avoiding the complications from operator mixing.

Due to conformal invariance, all the nontrivial information of the three-point function is contained in the x-independent coefficient $C_{i_1 i_2 i_3}$. It is natural to expect that $C_{i_1 i_2 i_3}$ is related to 3-strings interaction in pp-wave background. One such proposal was put forward in [3] and further analyzed in [24]. The proposal of [3] states that the matrix element of the lightcone hamiltonian is related to the coefficient of the three-point function in field theory via

$$\langle i|P^-|j,k\rangle = \mu(\Delta_i - \Delta_j - \Delta_k)C_{ijk} \tag{1.14}$$

in the leading order in λ'. Another proposal considered in [25] relates the ratio of the three-string amplitudes with those of the field theory three-point function coefficients

$$\frac{\langle\Phi_1|\langle\Phi_2|\langle\Phi_3|V\rangle}{\langle 0_1|\langle 0_2|\langle 0_3|V\rangle} = \frac{C_{123}}{C_{123}^{(\mathrm{vac})}}\,. \tag{1.15}$$

Here $\langle\Phi_1|\langle\Phi_2|\langle\Phi_3|V\rangle$ is the three-string scattering amplitude in the string field theory formalism, $\langle 0_1|\langle 0_2|\langle 0_3|V\rangle$ is the vacuum amplitude and V is the lightcone three-string vertex [27]. C_{123} (resp. $C_{123}^{(\mathrm{vac})}$) is the three-point function coefficient of the corresponding BMN operators (resp. of the "vacuum" operators (1.8)).

[1]The operators in (1.8), (1.9), (1.10) are already normalized such that (1.12) holds in free field theory.

All the tests in [3, 24, 25] were restricted to the free field value of C_{123}. In section 3, we will calculate the three-point functions in gauge theory to the first nontrivial order in λ'. According to (1.6), our results correspond to string theory in the subleading order in $\lambda' = 1/(\mu p^+ \alpha')^2$. In section 4, we will derive the corresponding three-string amplitudes and find perfect agreement with the field theory results to $O(\lambda')$. The string computation can be easily generalized to all orders in λ' and we obtain the exact form of the Neumann matrices explicitly. Our string theory results give an exact prediction for the field theory three-point functions.

2. Two-point function: normalization of BMN operators at order λ'

At the first order in λ', the normalization of $\mathcal{O}^J_{n,-n}$ in (1.10) has to be modified. To determine this normalization, we have to know precisely the two-point function to order λ'. The correlation functions of composite operators require UV regularization. We will use dimensional reduction to $D = 4 - 2\epsilon$ dimensions and work with Feynman rules in coordinate space. Since we work with planar diagrams, the group indices are trivial and the scalar propagator is given by

$$\Delta(x) = \frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}(x^2)^{1-\epsilon}} \,. \tag{2.1}$$

The scalar four-point interaction can be conveniently divided into F-terms and D-terms:

$$\mathcal{L}_F = -2g^2_{\rm YM}\, \mathrm{tr}\left([Z,\Phi][\bar\Phi,\bar Z] + [Z,\Psi][\bar\Psi,\bar Z] + [\Phi,\Psi][\bar\Psi,\bar\Phi]\right) , \tag{2.2}$$

$$\mathcal{L}_D = -g^2_{\rm YM}\, \mathrm{tr}\left([Z,\bar Z]^2 + 2[\Phi,\bar\Phi][Z,\bar Z] + 2[\Psi,\bar\Psi][Z,\bar Z] + 2[\Phi,\bar\Phi][\Psi,\bar\Psi]\right) . \tag{2.3}$$

At the planar level and to the first order in λ', by inspecting individual diagrams it is easy to see that only the F-term interactions (2.2) will contribute to the two and three-point functions; the D-term interactions, as well as the scalar self energy corrections and gluon exchanges between two scalars will have a vanishing net contribution. The cancellation in fact takes place to all orders in the genus expansion as was explained in [26, 3].

Taking into account one insertion of the vertices from $\mathcal{L}_F$, and combining with the free result, one obtains the two-point function in the large J limit

$$G_2(x) := \langle \bar{\mathcal{O}}^J_{n,-n}(0)\mathcal{O}^J_{n,-n}(x)\rangle = \Delta(x)^{J+2}(1 - 8\pi^2\delta \cdot I(x)) \,, \tag{2.4}$$

where

$$\delta = n^2\lambda' \tag{2.5}$$

is the anomalous dimension of the operator $\mathcal{O}^J_{n,-n}$, and $I(x)$ is the interaction integral with $\Delta(x)^2$ removed:

$$\begin{aligned} I(x) &:= \left(\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\right)^2 (x^2)^{2-2\epsilon} \int \frac{d^{4-2\epsilon}y}{(y^2)^{2-2\epsilon}(y-x)^{2(2-2\epsilon)}} \\ &= \frac{1}{8\pi^2}\left(\frac{1}{\epsilon} + \gamma + 1 + \log\pi + \log x^2 + O(\epsilon)\right) . \end{aligned} \tag{2.6}$$

We use a subtraction scheme which subtract the $1/\epsilon$ pole together with a finite part s

$$\frac{1}{\epsilon} + s \,. \tag{2.7}$$

To comply with the canonical form (1.12), the properly normalized $\mathcal{O}^J_{n,-n}$ is given by

$$\mathcal{O}^J_{n,-n} = \frac{1+\frac{\delta}{2}(\gamma+1-\log 4\pi - s)}{\sqrt{JN^{J+2}}} \sum_{l=0}^{J} \mathrm{tr}(\Phi Z^l \Psi Z^{J-l}) e^{\frac{2\pi i n l}{J}}. \tag{2.8}$$

The $\log x$ term on the right hand side of (2.6) was originally calculated in [1] and this was sufficient to extract the anomalous dimension. Here we have determined the scheme dependent finite part. For physical quantities, the scheme dependence must disappear. After we normalize the operators according to (1.12), all the correlation functions must be scheme independent. In the next section, we will calculate $C_{i_1 i_2 i_3}$.

3. Three-point function

In this section, we will calculate two simple examples of three-point functions

$$\begin{aligned} G_3(x_1,x_2) &:= \langle \bar{\mathcal{O}}^J_{n,-n}(0)\, \mathcal{O}^{J_1}_{0,0}(x_1)\, \mathcal{O}^{J_2}_{\mathrm{vac}}(x_2)\rangle\,, \\ \tilde{G}_3(x_1,x_2) &:= \langle \bar{\mathcal{O}}^J_{n,-n}(0)\, \mathcal{O}^{J_1}_{0}(x_1)\, \mathcal{O}^{J_2}_{0}(x_2)\rangle\,, \end{aligned} \tag{3.1}$$

to the first order in λ' and at the planar level. Here $J = J_1 + J_2$.

We first consider G_3. The free field theory diagrams are shown in figure 1 and the surviving interacting diagrams which arise from the F-term interactions are shown in figure 2. Our result is given by

$$G_3 = \frac{J_2}{N\sqrt{JJ_1J_2}}\left[1+\frac{\delta}{2}(\gamma+1-\log 4\pi - s)\right]\cdot \Delta(x_1)^{J_1+2}\Delta(x_2)^{J_2}(P_1+\lambda K(x_1,x_2)P_2)\,. \tag{3.2}$$

The factor $[1+\frac{\delta}{2}(\gamma+1-\log 4\pi - s)]/(N^{J+2}\sqrt{JJ_1J_2})$ arises from the normalizations of the operators; in addition summing over the loops of the planar diagram gives rise to a factor of N^{J+1}. Another factor of J_2 arises from inequivalent Wick contractions of $\mathcal{O}_{\mathrm{vac}}$ with the rest.

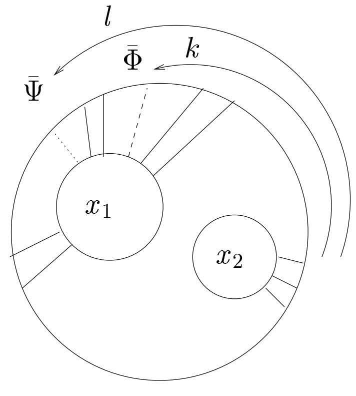


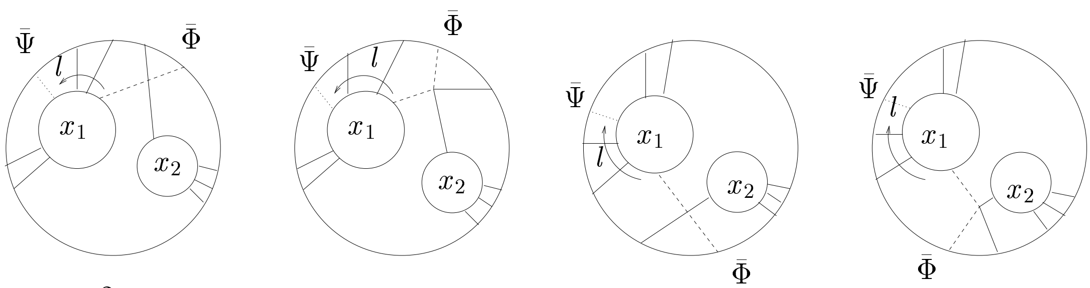
Figure 1: Free diagrams for G_3 contributing to P_1. The labels k and l count the Z-lines as indicated (for the diagram drawn above, $k = 2$, $l = 4$).

Each of the diagrams carries a phase factor which arises from the operator $\mathcal{O}_{n,-n}$ and depends on the relative position of Φ and Ψ. To obtain the total contribution, one has to sum over all possible positions of Φ and Ψ which amounts to inequivalent "electrostatic" diagrams in figures 1, 2, 3 and 4. In (3.2), P_1 and P_2 represent the total contributions of these phase factors for the free and interacting diagrams of figures 1 and 2 respectively. The contributions of diagrams in figures 3 and 4 will be shown to sum to zero. For P_1, we have

$$P_1 = \sum_{0\le k,l\le J_1} e^{\frac{2\pi i n}{J}(l-k)} \tag{3.3}$$

and in the large J limit

$$P_1 = \frac{J^2}{n^2\pi^2}\sin^2\left(\frac{n\pi J_1}{J}\right). \tag{3.4}$$



$2a$ $\qquad$ $2b$ $\qquad$ $2c$ $\qquad$ $2d$

Figure 2: Interacting diagrams for G_3 contributing to P_2. Diagrams $2a$ and $2c$ have positive signs. Diagrams $2b$ and $2d$ have negative signs.

The phase factor P_2 arises from the diagrams with one interaction vertex inserted (figure 2). We only need to take into account interaction vertices arising from the F-terms since as mentioned earlier, D-term scalar interactions, self energy corrections and gluon exchanges sum to zero. We find

$$\begin{aligned} P_2 &= 2(e^{-\frac{2\pi i n}{J}} - 1)\sum_{l=0}^{J_1} e^{-\frac{2\pi i n l}{J}} + 2(e^{\frac{2\pi i n}{J}} - 1)\sum_{l=0}^{J_1} e^{\frac{2\pi i n l}{J}} \\ &= \frac{J^2}{n^2\pi^2}\sin^2\left(\frac{n\pi J_1}{J}\right)\cdot\frac{-8\pi^2 n^2}{J^2}\,. \end{aligned} \tag{3.5}$$

The first term on the right hand side of (3.5) comes from the diagrams $2a$ and $2b$. The relative sign is easily seen from the commutators in $\mathcal{L}_F$. The second term in (3.5) comes from the diagrams $2c$ and $2d$, where the Φ interaction is now at the bottom. The multiplicative factors of 2 in (3.5) arise from summing the diagrams in figure 2 with Φ and Ψ exchanged.

The remaining F-terms diagram are shown in figures 3 and 4 and it is easy to see that there is a precise cancellation diagram by diagram between figure 3 and figure 4. For example, diagram $3a$ cancels diagram $4a$ as they have the same phase factor but opposite sign. Hence these classes of diagrams do not contribute to G_3.

Finally $K(x_1,x_2)$ is the interaction integral for diagrams 2 with $\Delta(x_1)\Delta(x_2)$ removed:

$$K(x_1,x_2) = \left(\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\right)^2 (x_1^2)^{1-\epsilon}(x_2^2)^{1-\epsilon}\int\frac{d^{4-2\epsilon}y}{(y^2)^{2-2\epsilon}(y-x_1)^{2(1-\epsilon)}(y-x_2)^{2(1-\epsilon)}} \tag{3.6}$$

Evaluating this integral, we obtain

$$K(x_1,x_2) = \frac{1}{16\pi^2}\left(\frac{1}{\epsilon} + \gamma + 2 + \log\pi + \log\frac{x_1^2 x_2^2}{x_{12}^2} + O(\epsilon)\right). \tag{3.7}$$

Using our subtraction (2.7), we obtain

$$G_3(x_1,x_2) = \frac{C_{123}}{(4\pi^2 x_1^2)^{J_1+2+\frac{\delta}{2}}(4\pi^2 x_2^2)^{J_2+\frac{\delta}{2}}(4\pi^2 x_{12}^2)^{-\frac{\delta}{2}}}\,, \tag{3.8}$$

where the three-point function coefficient is

$$C_{123} = \frac{J^{3/2}J_2^{1/2}}{NJ_1^{1/2}}\frac{1}{n^2\pi^2}\sin^2\left(\frac{n\pi J_1}{J}\right)\left(1 - \frac{\lambda' n^2}{2}\right). \tag{3.9}$$

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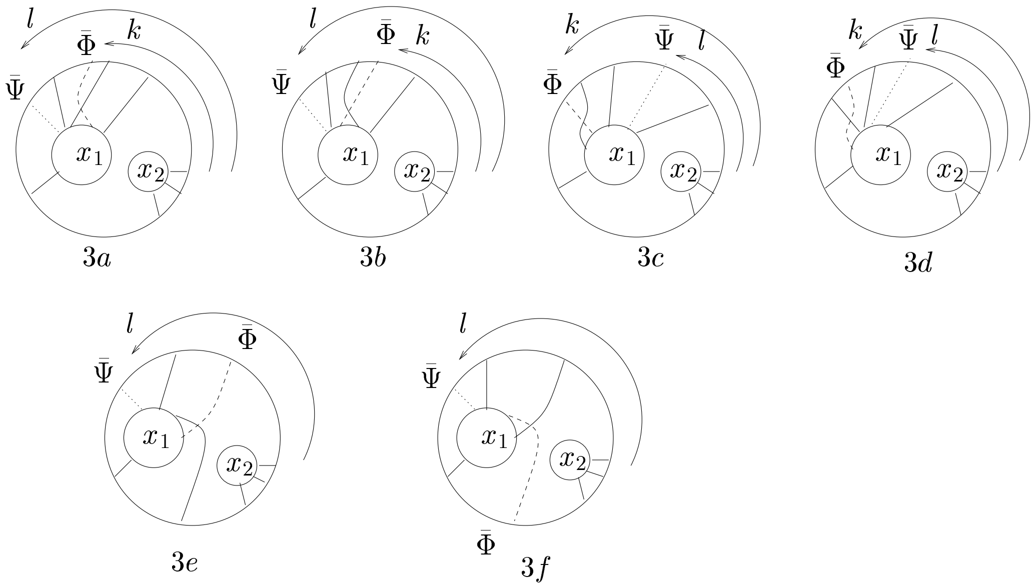

Figure 3: Interacting diagrams for G_3. All diagrams come with a positive sign.

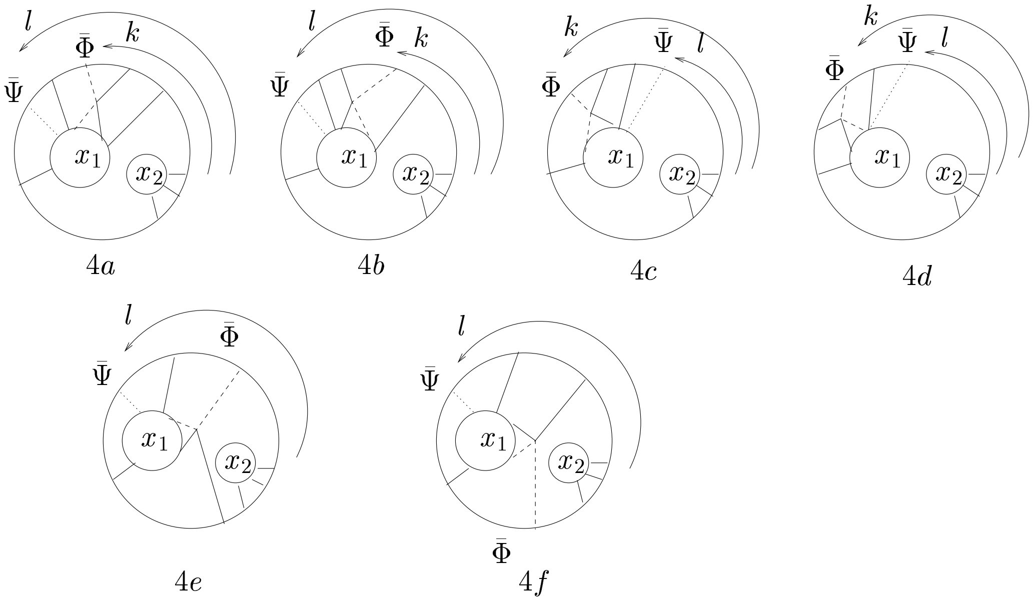

Figure 4: Interacting diagrams for G_3. All diagrams come with a negative sign and precisely cancel those in figure 3.

This is one of the main result of this paper and, as anticipated, it is scheme independent.

We now consider our second example of three-point function $\tilde{G}_3$. Considerations similar to the above lead us to the following expression

$$\tilde{G}_3 = \frac{1}{N\sqrt{J}}\left[1+\frac{\delta}{2}(\gamma+1-\log 4\pi - s)\right]\cdot\Delta(x_1)^{J_1+1}\Delta(x_2)^{J_2+1}(P_5+\lambda K(x_1,x_2)P_6)\,. \quad (3.10)$$

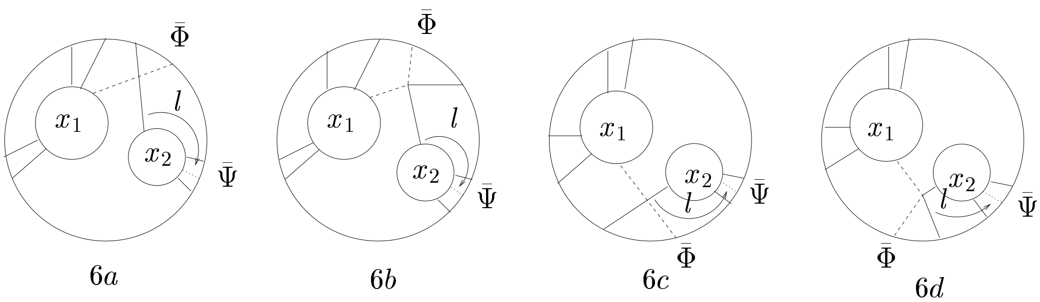

Figure 6: Interacting diagrams for $\tilde{G}_3$ contributing to P_6. Diagrams $6a$ and $6c$ have positive signs. Diagrams $6b$ and $6d$ have negative signs.

The only nontrivial difference from the previous case is encoded in the factors P_5 and P_6. The factor P_5 is the sum of phase factors from the free diagrams (see figure 5):

$$P_5 = \sum_{k=0}^{J_1} \sum_{l=0}^{J_2} e^{\frac{2\pi i(k+l)n}{J}} = -\frac{J^2}{n^2\pi^2} \sin^2\left(\frac{n\pi J_1}{J}\right). \tag{3.11}$$

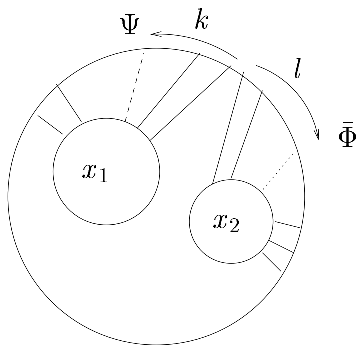

Figure 5: Free diagrams for $\tilde{G}_3$ contributing to P_5.

P_6 is the contribution of the phase factors from the interacting diagrams shown in figure 6,

$$\begin{aligned} P_6 &= 2\sum_{l=0}^{J_2-1} e^{\frac{2\pi inl}{J}}(1 - e^{\frac{2\pi in}{J}}) + 2\sum_{l=0}^{J_1-1} e^{\frac{2\pi inl}{J}}(1 - e^{\frac{2\pi in}{J}}) \\ &= -\frac{J^2}{n^2\pi^2}\sin^2\left(\frac{n\pi J_1}{J}\right) \cdot \frac{-8\pi^2 n^2}{J^2}. \end{aligned} \tag{3.12}$$

We remark that the classes of diagrams similar to figure 3 and figure 4 cancel each other identically for the same reason as before.

Finally we obtain

$$\tilde{G}_3(x_1, x_2) = \frac{\tilde{C}_{123}}{(4\pi^2 x_1^2)^{J_1+1+\frac{\delta}{2}} (4\pi^2 x_2^2)^{J_2+1+\frac{\delta}{2}} (4\pi^2 x_{12}^2)^{-\frac{\delta}{2}}}, \tag{3.13}$$

where the three-point function coefficient is

$$\tilde{C}_{123} = -\frac{J^{3/2}}{N}\frac{1}{n^2\pi^2}\sin^2\left(\frac{n\pi J_1}{J}\right)\left(1 - \frac{\lambda' n^2}{2}\right). \tag{3.14}$$

4. Comparison with string theory

Last week, using the formalism of [28, 27], Huang [25] calculated the ratio on the LHS of (1.15) in the large $\mu p^+ \alpha'$ limit and found agreement with the free field theory expression. In this section, we will generalize his computation of the string amplitudes to all orders in $1/(\mu p^+ \alpha')^2$ and compare these new results with the field theory expressions derived in the last section. We find perfect agreement to $O(\lambda')$.

In the pp-wave background, the cubic string vertex is given by $|V\rangle = E_a E_b |0\rangle$, where E_a and E_b are the bosonic and fermionic operators. For our purposes, we need only the expression for E_a [27],

$$E_a \sim \exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty} a^{\dagger}_{m(r)}\overline{N}^{(rs)}_{mn}a^{\dagger}_{n(s)}\right], \tag{4.1}$$

where

$$\overline{N}^{(rs)}_{mn} = \delta^{rs}\delta_{mn} - 2\sqrt{\omega_{m(r)}\omega_{n(s)}}(X^{(r)\mathrm{T}}\Gamma_a^{-1}X^{(s)})_{mn} \tag{4.2}$$

are the Neumann matrices;

$$\omega_{n(r)} = \sqrt{n^2 + (\mu\alpha_{(r)})^2}\,, \qquad \alpha_{(r)} := \alpha' p^{+}_{(r)} \tag{4.3}$$

are the oscillation frequencies of the r-th string, and the matrix Γ_a is given by

$$(\Gamma_a)_{mn} = \sum_{r=1}^{3}\sum_{p=-\infty}^{\infty}\omega_{p(r)}X^{(r)}_{mp}X^{(r)}_{np}\,. \tag{4.4}$$

The matrices $X^{(r)}$ arise from the overlapping integrals in string field theory, they were computed in [27] and are independent of μ. Thus the whole μ dependence is concentrated in the frequencies $\omega_{m(r)}$.

To the first order in $1/(\mu\alpha' p^+)^2 = \lambda'$, we have

$$\begin{aligned}
\omega_{n(3)} &= \mu\alpha' p^+\left(1 + \frac{\lambda' n^2}{2}\right),\\
\omega_{n(1)} &= \kappa\mu\alpha' p^+\left(1 + \frac{\lambda' n^2}{2\kappa^2}\right),\\
\omega_{n(2)} &= (1-\kappa)\mu\alpha' p^+\left(1 + \frac{\lambda' n^2}{2(1-\kappa)^2}\right),
\end{aligned} \tag{4.5}$$

where $p^+ = p^+_{(3)}$. Here κ is the ratio of the lightcone momentum of string 1 and 3,

$$\kappa \equiv \kappa_1 := \frac{p^+_{(1)}}{p^+} = \frac{J_1}{J}\,, \qquad \kappa_2 := \frac{J_2}{J} = 1-\kappa\,. \tag{4.6}$$

Our next goal is to compute the matrix Γ_a. The leading order expression in the large $\mu\alpha' p^+$ limit was computed in [25] and reads

$$(\Gamma_a^{(0)})_{mn} = 2\mu\alpha' p^+\delta_{mn}\,. \tag{4.7}$$

At the next order in λ', we find

$$(\Gamma_a)_{mn} = (\Gamma_a^{(0)})_{mn} + \frac{\lambda'}{2}\sum_l\left[\frac{l^2}{\kappa_1^2}\omega^{(0)}_{l(1)}X^{(1)}_{ml}X^{(1)}_{nl} + (1\leftrightarrow 2)\right] + \frac{\lambda' n^2}{4}(\Gamma_a^{(0)})_{mn}\,, \tag{4.8}$$

where $\omega^{(0)}_{l(r)} = \kappa_r \mu\alpha' p^+$ are the frequencies in the lowest order in λ'. Using the explicit expressions for $X^{(r)}_{ml}$ and summation formulae from [28, appendix D], we obtain

$$(\Gamma_a)_{mn} = (\Gamma^{(0)}_a)_{mn} + \mu\alpha' p^+ \lambda' n^2 \delta_{mn} = (\Gamma^{(0)}_a)_{mn} \left(1 + \frac{\lambda' n^2}{2}\right). \tag{4.9}$$

Using this result, we derive the expression for the following elements of the Neumann matrices,

$$\overline{N}^{(3r)}_{n0} = [\overline{N}^{(3r)}_{n0}]^{(0)} \left(1 - \frac{\lambda' n^2}{4}\right), \qquad r = 1, 2, \tag{4.10}$$

which will be needed for the analysis below. The novelty in our results (4.9) and (4.10) lies in the second terms on the RHS, which are the $O(\lambda')$ corrections to the zeroth order results of [25] and

$$[\overline{N}^{(3r)}_{mn}]^{(0)} = -\sqrt{\kappa_r}(X^{(3)T} X^{(r)})_{mn}\,. \tag{4.11}$$

Now we consider the string scattering amplitudes which correspond to the field theory three-point functions G_3 and $\tilde{G}_3$. We have on the field theory side

$$\frac{C_{123}}{C^{(\mathrm{vac})}_{123}} = \frac{J}{J_1}\frac{1}{n^2\pi^2}\sin^2\left(\frac{n\pi J_1}{J}\right)\cdot\left(1 - \frac{\lambda' n^2}{2}\right), \tag{4.12}$$

$$\frac{\tilde{C}_{123}}{C^{(\mathrm{vac})}_{123}} = -\frac{J}{\sqrt{J_1 J_2}}\frac{1}{n^2\pi^2}\sin^2\left(\frac{n\pi J_1}{J}\right)\cdot\left(1 - \frac{\lambda' n^2}{2}\right), \tag{4.13}$$

where $C^{(\mathrm{vac})}_{123} = \sqrt{J J_1 J_2}/N$.

On the string side, we have for the first process (corresponding to (4.12)),

$$\frac{\langle\Phi_1|\langle\Phi_2|\langle\Phi_3|V\rangle}{\langle 0_1|\langle 0_2|\langle 0_3|V\rangle} = \frac{1}{4}\left(\overline{N}^{(31)}_{n0} - \overline{N}^{(31)}_{-n0}\right)^2 \tag{4.14}$$

$$= \frac{1}{4}\left([\overline{N}^{(31)}_{n0} - \overline{N}^{(31)}_{-n0}]^{(0)}\right)^2\cdot\left(1 - \frac{\lambda' n^2}{2}\right)$$

$$= \frac{J}{J_1}\frac{1}{n^2\pi^2}\sin^2\left(\frac{n\pi J_1}{J}\right)\cdot\left(1 - \frac{\lambda' n^2}{2}\right) \tag{4.15}$$

which is in perfect agreement with (4.12). Similarly for the second string scattering (corresponding to (4.13)), we obtain (cf. [25])

$$\frac{\langle\Phi_1|\langle\Phi_2|\langle\Phi_3|V\rangle}{\langle 0_1|\langle 0_2|\langle 0_3|V\rangle} = \frac{1}{2}\overline{N}^{(31)}_{n0}\overline{N}^{(32)}_{n0} \tag{4.16}$$

$$= \frac{1}{2}\left[\overline{N}^{(31)}_{n0}\overline{N}^{(32)}_{n0}\right]^{(0)}\cdot\left(1 - \frac{\lambda' n^2}{2}\right)$$

$$= -\frac{J}{\sqrt{J_1 J_2}}\sin^2\left(\frac{n\pi J_1}{J}\right)\cdot\left(1 - \frac{\lambda' n^2}{2}\right) \tag{4.17}$$

which is again in agreement with our field theory result.

Finally we generalize our string computations to all orders in λ'. To do this, we note that using the formulae in the appendix, it is easy to derive

$$(\Gamma_a)_{mn} = 2\mu\alpha' p^+\sqrt{1 + \lambda' n^2}\,\delta_{mn} = 2\omega_{n(3)}\delta_{mn}. \tag{4.18}$$

This was obtained by substituting the all-orders expansion of the square root in $\omega_{n(r)} = \sqrt{n^2 + (\mu\alpha_{(r)})^2}$ in the large μ limit into (4.4). The resulting expression involves an infinite sum arising from the multiplication of matrices of infinite dimension. In order to work with well-defined expressions, the sum has to be regularized. As standard in string theory, we use the zeta function regularization, as outlined in the appendix. It then follow from (4.2) that

$$\overline{N}^{(3r)}_{nm} = [\overline{N}^{(3r)}_{nm}]^{(0)} \left(\frac{1+\lambda' m^2/\kappa_r^2}{1+\lambda' n^2}\right)^{1/4}, \qquad r = 1, 2\,. \tag{4.19}$$

The results (4.18), (4.19) generalize (4.9), (4.10) to all orders in λ'. Equation (4.18) was derived by resumming all-orders expansions in the large μ (small λ') limit. These expansions contained only integer powers of λ'. Our analysis is complete (in the sense that it does not miss any terms) in the vicinity of $\lambda' = 0$ and it cannot be analytically extrapolated to the opposite regime of $\lambda' = \infty$. Even though our result (4.18) has a closed analytic form, it is interpreted as an asymptotic expansion[2] in powers of λ'. This allows us to compute the corresponding three-string amplitudes to all orders in small λ'. Note that the direct use of (4.18) to the opposite regime of large λ' is not allowed, and in fact the Neumann matrices for $\lambda' = \infty$ are known and cannot be obtained from our formulae.

Finally, this leads to the field theory predictions

$$\frac{C_{123}}{C_{123}^{(\mathrm{vac})}} = \frac{J}{J_1}\frac{1}{n^2\pi^2}\sin^2\left(\frac{n\pi J_1}{J}\right)\cdot(1+\lambda' n^2)^{-1/2}\,, \tag{4.20}$$

$$\frac{\tilde{C}_{123}}{C_{123}^{(\mathrm{vac})}} = -\frac{J}{\sqrt{J_1 J_2}}\frac{1}{n^2\pi^2}\sin^2\left(\frac{n\pi J_1}{J}\right)\cdot(1+\lambda' n^2)^{-1/2}\,. \tag{4.21}$$

It would be interesting to verify this all-orders prediction from the field theory point of view.

A. Summation formulae

We note the useful identity [29]

$$\sum_{l=-\infty}^{\infty} (-1)^l \frac{e^{ily}}{l+\beta} = \frac{\pi}{\sin(\beta\pi)} e^{-i\beta y}\,, \qquad -\pi < y < \pi\,, \tag{A.1}$$

from which one can derive

$$\sum_{l=1}^{\infty} \frac{(-1)^l}{l^2-\beta^2}\cos(ly) = \frac{1}{2\beta^2} - \frac{\pi}{2\beta\sin(\beta\pi)}\cos(\beta y)\,, \tag{A.2}$$

and formally

$$\sum_{l=1}^{\infty} \frac{l^{2p}}{l^2-\beta^2} = -\frac{\pi}{2}\beta^{2p-1}\cot(\pi\beta)\,, \qquad p \geq 1\,, \tag{A.3}$$

[2] After this work appeared, it was argued in [30] that terms proportional to $(\lambda')^{3/2}$ will typically appear on the string side, which is puzzling from the field theory point of view. We do not see such terms in our analysis.

$$\sum_{l=1}^{\infty} \frac{l^{2p}}{(l^2-\beta^2)(l^2-\gamma^2)} = -\frac{\pi}{2(\beta^2-\gamma^2)} \left(\beta^{2p-1}\cot(\beta\pi) - \gamma^{2p-1}\cot(\gamma\pi)\right), \qquad p \geq 1\,, \tag{A.4}$$

$$\sum_{l=1}^{\infty} \frac{l^{2p}}{(l^2-\beta^2)^2} = \frac{\pi}{4}\left[\pi\beta^{2p-2}\csc^2(\beta\pi) - (2p-1)\beta^{2p-3}\cot(\beta\pi)\right], \qquad p \geq 1\,. \tag{A.5}$$

The formulae (A.3), (A.4) and (A.5) are understood by an appropriate analytic continuation using zeta function regularization. In more details, define

$$f(s) := \sum_{l=1}^{\infty} \frac{1}{l^s}\frac{1}{l^2-\beta^2} = \sum_{k=0}^{\infty} \beta^{2k}\zeta(s+2+2k)\,. \tag{A.6}$$

Then we obtain (A.3)

$$f(-2p) = -\frac{\pi}{2}\beta^{2p-1}\cot(\pi\beta)\,. \tag{A.7}$$

Equations (A.4) and (A.5) follow immediately.

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JHEP06(2002)011

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